ON SOME EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. In this paper we have discussed a boundary value problem involving Caputo nonlinear fractional integro-differential equations of order $0 < \alpha \leq 1$ *and* $0 < \beta \leq 1$ *with boundary conditions of the form* $x(0) = x(1) = 0$. *We have proved some new existence and uniqueness results by using the fixed point theory. In particular, we have used the Banach contraction mapping principle and Krasnoselskii's fixed point theorem under some weak conditions. The results proved are supported by means of a couple of examples.*

Keywords: Riemann-Liouville fractional derivative, Caputo fractional differential equation, Banach contraction principle, Krasnoselskii's fixed point theorem

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1. INTRODUCTION:

After the wide and successful applicability of the theory of differential equations in the fields of Applied Mathematics, Mathematical Physics, Chemical Sciences, Biological Sciences, Engineering and Technology, etc., the theory of fractional calculus has attracted the attention of many researchers because of the applicability of the derivatives and integrals of the fractional order with the corresponding initial and boundary conditions. Besides all the fields of sciences and technology as mentioned earlier, the theory of fractional calculus is being applied to Fluid Dynamics, Electromagnetism, Viscoelasticity, the Analysis of the Feedback Amplifiers and Capacitors, etc. In last few decades, many of the researchers have pointed out that the fractional order differentials and integrals are of special importance in order to describe the viscoelastic properties of the real materials like polymers. In this paper, we have considered the existence and uniqueness of solutions for the following problem:

$$D^{\alpha}D^{\beta}x(t) = f\big(t, x(t), \phi x(t), \psi x(t)\big), t\epsilon[0,1] \quad . \; . \; . \; (1)$$
$$x(0) = x(1) = 0$$

where $0 < \alpha \leq 1, 0 < \beta \leq 1$, D^{α}, D^{β}, are the Caputo fractional derivatives of order α , β, $f : [0,1] \times R^3 \longrightarrow R$ is a continuous function, and

$$\phi x(t) = \int_0^t \lambda(t,s)x(s)ds \quad . \; . \; . \quad (2)$$

$$\psi x(t) = \int_0^t \delta(t,s)x(s)ds \quad . \; . \; . \quad (3)$$

where λ, δ:[0,1]×[0, 1]→[0,+∞) with

$$\phi^* = Sup_{t\in[0,1]}\left|\int_0^t \lambda(t,s)ds\right| < \infty \quad . \; . \; . \; (4)$$

$$\psi^* = Sup_{t\in[0,1]}\left|\int_0^t \delta(t,s)ds\right| < \infty \quad . \; . \; . \; (5)$$

Before proving the existence of the solution to the boundary value problem (1–5), we will take a review of the basic definitions and the notions required for the understanding of these results in the next section.

2. A REVIEW OF PRELIMINARY CONCEPTS AND RESULTS

Leibnitz discussed the fractional derivative of order 1.5 in his notes to L'Hospital back in the year 1695. Joseph Fourier in 1822 gave an expression for a fractional order derivative[1] obtained from the Fourier integral representation of a function in the form

$$\frac{d^u[f(x)]}{dx^u} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)d\alpha \int_{-\infty}^{\infty} p^u \cos\left[p(x-a) + \frac{u\pi}{2}\right]dp$$

The first major study of fractional calculus was made by Liouville in 1833 who gave two definitions of fractional order derivatives as follows. The arbitrary derivative $D^{\,v}$of order v of a function $f(x)$ having power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \qquad Re(a_n) > 0$$

is given by

$$D^{\nu}[f(x)] = \sum_{n=0}^{\infty} c_n {a_n}^{\nu} e^{a_n x}$$

Because of the restrictions on the function $f(x)$, Liouville [2] gave his second definition involving the gamma function in the form

$$D^{\nu} x^{-a} = \frac{(-1)^{\nu}}{\Gamma(a)} \int_0^{\infty} u^{a+\nu-1} e^{-xu}\, du$$

$$= \frac{(-1)^{\nu}\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu}, \qquad a > 0$$

The second definition given by Liouville is too narrow as it applies only to the functions of the form $f(x) = x^{-a}$, $a > 0$. Using the generalization of the Taylor series expansion of a function, Bernhard Riemann[2] defined the fractional derivative of order ν as

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt + \psi(x)$$

where Riemann added a complementary function $\psi(x)$ as the lower limit of the integration c was unclear. The difficulty in the applicability of the Riemann's definition was pointed out by A. Cayley since it was unclear what will be the meaning of the complementary function $\psi(x)$ if $\psi(x)$ has an infinite arbitrary constants.

As a modern approach towards defining the fractional order derivative, we use the Riemann-Liouville definition of the fractional order derivative of a function $f(t)$ defined on the closed interval $[a, t]$ and having the $(m+1)$th continuous derivative $f^{(m+1)}(t)$. The Riemann-Liouville derivative [2, 3, 4, 5] of fractional order α is given by

$$D_t^p[f(t)] = \frac{\mathrm{d}^{(m+1)}}{\mathrm{dt}^{(m+1)}} \int_a^t (t-\tau)^{(m-p)} f(\tau)\, d\tau$$

where $m \le p < m+1$. The initial value problem involving Riemann-Liouville fractional derivative are practically not useful as there is no physical interpretation of such types of initial conditions.

The general approach suggested by M. Caputo is useful for the formulation of initial value problems involving the fractional differential and integral equations. Caputo's definition [2] of the derivative of fractional order α is given by

$${}_a^C D_t^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau$$

It is clear that as $\alpha \to n$, the the Caputo's definition becomes the conventional definition of the nth order $f^{(n)}(t)$. The main advantage of Caputo's definition is that the initial conditions for the fractional differential equations with Caputo derivatives take on the same form as the integer order differential equations. We make a slight change in the notation for Caputo fractional derivative and define the Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function f as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau$$

where n is a positive integer, $0 \le n-1 < \alpha < n$ and $t > 0$. The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

In [6], by the application of Krasnoselskii Fixed point theorem, Agarwal et al. have proved the existence of at least one solution to the initial value problem of fractional neural functional differential equation given by

$$D_C^{\alpha}[x(t) - f(t, x_t)] = f(t, x_t), \qquad t \in (t_0, \infty), t_0 \ge 0$$

$$x_{t_0} = \phi$$

where D_C^{α} is the Caputo fractional derivative of order α, $0 < \alpha < 1$, f and g are functions defined on $[t_0, 1) \times C([-r, 0],\ R^n) \to R^n$, $\phi \in C([-r, 0],\ R^n)$, $a > 0$. In [7], Fang Li has proved the existence and uniqueness of mild solutions in a Banach space X for the fractional differential equation of the form

$$\frac{d^q}{dt^q}[x(t)] = -A\, x(t) + f\big(t, x(t), Gx(t)\big),$$
$$t \in [0, T]$$

under the conditions

$$x(0) + g(x) = x_0, \qquad 0 < q < 1, \qquad T > 0$$

where it is assumed that $-A$ generates an analytic semi-group $\{s_t\}_{t \ge 0}$ of uniformly bounded linear operators on the space X, the operator $Gx(t)$ is defined by

$$G\,x(t) = \int_0^t k(t,s)\,x(s)\,ds$$

where K is a positive function defined on the set $D = \{(t,\ s) \in R2 : 0 \leq s \leq t \leq T\}$ and

$$G^* = \sup_{t\in[0,\ T]} \int_0^t k(t,s)\ ds < \infty$$

Ahmad et al.[8, 9], have obtained the solutions of the integrodifferential equations with non-local four point and strip multipoint boundary conditions. Wang et al. in [10] have established the conditions for the uniqueness and existence of the positive solutions of the fractional integrodifferential equation

$$D^\alpha u(t) + f\big(t, u(t), Tu(t), Su(t)\big) = 0, \qquad 0 < t < 1$$

under the boundary conditions given by

$$u(0) = u_0, u^{'}(0) = b_1, \dots, u^{(n-3)}(0) = b_{n-3}, u^{(n-2)}(0) = b_{n-2}, u^{(n-1)}(0) = b_{n-1}$$

where $n-1 < \alpha \leq n,\ 0 \leq \mu < n-1,\ n \geq 3,\ bi \geq 0\ (i = 1,\ 2, \dots,\ n-3,\ n-2,\ n-1),\quad D^\alpha$ being the Caputo fractional derivative of order α, f is a continuous function from $[0,\ 1] \times R3 + R_+^3 \to R_+$, T and S are defined by

$$(Tx)(t) = \int_0^1 K(t,s)\,x(s)\,ds, \quad (Sx)(t) = \int_0^1 H(t,s)\,x(s)\,ds$$

$$K^* = \sup_{t\in[0,\ 1]} \int_0^t K(t,s)\ ds,$$

$$H^* = \sup_{t\in[0,\ 1]} \int_0^t H(t,s)\ ds$$

where $K \in C(D,\ R^+),\ H \in C([0,1] \times\ [0,1],\ R^+)$.

Many more others like Hilal et al. [11, 12] also have obtained the results stating the existence and uniqueness of the solutions of the fractional integro-differential equations under different boundary conditions. In [13], A. Bragdi et al. obtained the solution of the BVP given by

$$D^\alpha\big(D^\beta\big)u(t) = f\big(t, u(t), \phi u(t), \psi u(t)\big)$$

under the boundary conditions given by

$$u(1) = u(0) = u^{'}(0) = 0$$

where it is assumed that $1 < \alpha \leq 2,\ 0 < \beta \leq 1$, $f : I \times R^3 \to R,\ I = [0,\ 1]$, the function f is continuous and

$$\phi(u)(t) = \int_0^t \gamma(t,s)u(s)\ ds, \quad \psi(u)(t) = \int_0^t \lambda(t,s)u(s)\ ds$$

$$sup \int_0^1 \lambda(t,s)\ ds < \infty, \qquad sup \int_0^1 \gamma(t,s)\ ds < \infty$$

where $\gamma, \lambda : I \times I \to [0,1)$

In [14], Ibnelazyz L et al. have explored the existence and uniqueness for a nonlinear fractional integro-differential equations with integral and anti-periodic boundary conditions where the existence is proved by means of Krasnoselskii's fixed point theorem and the uniqueness of solutions is established via the Banach's contraction principle. In [15], M. J. Mardanov et al. have obtained the unique solution for the BVP

$$D_{0^+}^\alpha\, x(t) = f\big(t, x(t), \phi\, x(t), \psi\, x(t)\big), t \in [0,T]$$

under the boundary conditions described by

$$Ax(0) = \int_0^T n(t)x(t)dt = C$$

where $0 < \alpha < 1$, $D_{0^+}^\alpha$ is the Caputo fractional derivative of order $\alpha, A \in R^{n\times n}$, $n(t)$ is a function $[0,T] \to R^{n\times n}$. The other terms are defined by

$$N = A + \int_0^T n(t)dt\,, \qquad det(N) \neq 0$$

$$\phi(x)(t) = \int_0^t \lambda(t,s)x(s)ds\,,\ \ \psi(x)(t) = \int_0^T \gamma(t,s)x(s)ds$$

where

μ, λ: $[0,T] \times [0,T] \to\ R^{n\times n}$,

$$\mu_0 = max \left||\mu(t,s)|\right|, \lambda_0 = max \left||\mu(t,s)|\right|, \quad t,s \in [0,T]$$

3. MAIN RESULTS

In this section we will prove the existence of the solution of the initial value problem (1-5). First we mention some of the important results required.

Theorem 1. [16] *Let Ω be a closed, convex, and bounded nonempty subset of a Banach space X. Let A and B be two operators such that*
(i) $Ax + By \in \Omega$ whenever $x, y \in \Omega$
(ii) A is compact and continuous
(iii) B is a contraction mapping
Then, there exists $z \in \Omega$ such that $z = Az + Bz$.

Theorem 2. [3] *Let $\alpha, \beta \geq 0$. Then the following relation hold*

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$$

Theorem 3. [3] *Let n be a positive integer and $n-1 < \alpha < n$. If f is a continuous function then we have*

$$I^\alpha D^\alpha[f(t)] = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}$$

Theorem 4. *Let $f \in C([0,1], R)$ then the unique solution to the initial value Problem*

$$D^\alpha D^\beta x(t) = f(t), \qquad t\epsilon[0,1]$$

$$x(0) = x(1) = 0$$

is given by

$$x(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau) d\tau - \frac{t^\beta}{\Gamma(\alpha+\beta)} \int_0^1 (1-\tau)^{\alpha+\beta-1} f(\tau) d\tau$$

Proof: By applying theorem 3, we have

$$D^\beta x(t) = I^\alpha f(t) + a_0$$

$$x(t) = I^{\alpha+\beta} f(t) + I^\beta a_0 + a_1$$

where $a_0, a_1 \in R$.

Hence

$$x(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau) d\tau + \frac{t^\beta}{\Gamma(\beta+1)} a_0 + a_1$$

and by using the condition $x(0) = 0$, we obtain $a_1 = 0$ and by using $x(1) = 0$ we get

$$a_0 = -\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} f(\tau) d\tau$$

By substituting the value of a_0, we get

$$x(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau) d\tau - \frac{t^\beta}{\Gamma(\alpha+\beta)} \int_0^1 (1-\tau)^{\alpha+\beta-1} f(\tau) d\tau.$$

The converse can be easily verified by direct computations. ■

Theorem 5. (Main Result: Existence of the Solution)

Let X be the Banach space of all continuous function from $[0,1] \to R$ induced with the norm

$\|y\| = sup_{t\in[0,1]}\{|y(t)|: t \in [0,1]$ and $\|y\|_\mu = sup_{t\in[0,1]} \left(\frac{y(t)}{e^{\mu t}}\right)$

where $\mu > \left(1 + \phi^* + \psi^*)\Gamma(\alpha+\beta)\right)\|\sigma\|, \sigma \in C([0,1]; [0,\infty))$.

Suppose that

1. $\left|f\left(t, x_1, x_2, x_{3,}\right) - f\left(t, y_1, y_2, y_{3,}\right)\right|$

$\leq \sigma(t)(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|)$

for all $t \in [0,1]$ and $x_1, x_2, x_{3,} y_1, y_2, y_3 \in R$.

2. $|f(t, x, y, z)| \leq k(t)$

$\forall\ (t, x, y, z) \in [0,1] \times R^3, \quad k \in C([0,1]; R^+)$

Then the initial value problem (1-5) has at least one solution.

Proof: Consider an ϵ-sphere $B_\epsilon = \left\{y \in X: \|y\|_\mu \leq \epsilon\right\}$ with

$$\epsilon \geq \frac{\|k\|}{\mu}\left(\frac{e^\mu - 1}{\Gamma(\alpha+\beta)}\right) + \frac{1}{\Gamma(\alpha+\beta)}$$

We define two operators A and B on B_ϵ by the relations

$$A_{x(t)} = \frac{1}{\Gamma(\alpha+\beta)} \times$$

$$\int_0^t (t-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big)\, d\tau$$

$$B_{y(t)} = -\frac{t^\beta}{\Gamma(\alpha+\beta)} \times$$

$$\int_0^1 (1-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big)\, d\tau$$

For $x, y \in B_\epsilon$, we have

$$\left\|A_{x(t)}\right\|_\mu$$
$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \left| \frac{1}{\Gamma(\alpha+\beta)} \times \int_0^t (t-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big)\, d\tau \right|$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} |k(\tau)| d\tau$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{1}{\Gamma(\alpha+\beta)} \times$$

$$\int_0^t (t-\tau)^{\alpha+\beta-1} |k(\tau)| \frac{e^{\mu\tau}}{e^{\mu\tau}} d\tau$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} e^{\mu\tau} d\tau$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)} \int_0^t e^{\mu\tau} d\tau$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)} \frac{e^{\mu t}-1}{\mu}$$

$$\leq sup_{t\in[0,1]} \frac{1}{\mu} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)} \frac{e^{\mu t}-1}{e^{\mu t}}$$

$$\leq \frac{1}{\mu} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)}$$

Also, we have

$$\left\|B_{y(t)}\right\|_\mu$$
$$\leq\; sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \left| -\frac{t^\beta}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big)\, d\tau \right|$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \left| \frac{t^\beta}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big) d\tau \right|$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-\tau)^{\alpha+\beta-1} k(\tau) \frac{e^{\mu\tau}}{e^{\mu\tau}} d\tau$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)} \int_0^1 (1-\tau)^{\alpha+\beta-1} e^{\mu\tau} d\tau$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)} \int_0^1 e^{\mu\tau} d\tau$$

$$\leq sup_{t\in[0,1]} \frac{1}{e^{\mu t}} \frac{\|k\|_\mu}{\Gamma(\alpha+\beta)} \frac{e^{\mu}-1}{\mu}$$

$$\leq \frac{\|k\|_\mu}{\mu} \frac{e^{\mu}-1}{\Gamma(\alpha+\beta)}$$

Therefore

$$\left\|A_{x(t)} + B_{x(t)}\right\|_\mu \leq \frac{\|k\|_\mu}{\mu} \left[\frac{e^{\mu}-1}{\Gamma(\alpha+\beta)} + \frac{1}{\Gamma(\alpha+\beta)}\right]$$

This proves that $xA + yB \in B_\epsilon$.

Now we prove that A is a contraction mapping. For $x, y \in B_\epsilon$, we have

$$\left\|A_{y(t)} - A_{x(t)}\right\|_\mu$$

$$\leq \sup \frac{1}{\Gamma(\alpha+\beta) e^{\mu t}} \int_0^t (t-\tau)^{\alpha+\beta-1}$$

$$\times \left| f\big(\tau, y(\tau), \phi y(\tau), \psi y(\tau)\big) - f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big) \right| d\tau$$

$$\leq \sup \frac{1}{\Gamma(\alpha+\beta) e^{\mu t}} \int_0^t (t-\tau)^{\alpha+\beta-1}$$

$$\times \sigma(\tau)[\, |y(\tau) - x(\tau)| + |\phi y(\tau) - \phi x(\tau)| + |\psi y(\tau) - \psi x(\tau)|]\, d\tau$$

$$\leq sup_{t\in[0,1]} \frac{\|\sigma\|}{\Gamma(\alpha+\beta) e^{\mu t}} \int_0^t e^{\mu\tau} \left[\|y-x\|_\mu + \phi^* \|y-x\|_\mu + \psi^* \|y-x\|_\mu\right] d\tau$$

$$\leq sup_{t\in[0,1]} \frac{(1+\phi^*+\psi^*)\|\sigma\|}{\Gamma(\alpha+\beta)} \frac{e^{\mu t}-1}{e^{\mu t}} \left[\|y-x\|_\mu\right]$$

$$\leq sup_{t\in[0,1]} \frac{(1+\phi^*+\psi^*)\|\sigma\|}{\Gamma(\alpha+\beta)} \left[\|y-x\|_{\mu}\right]$$

From the definition of the new norm, we conclude that A is a contraction mapping. Moreover, the continuity of the function f implies that B is compact and continuous. Also B is uniformly bounded on B_{ϵ} since

$$\left\|B_{y(t)}\right\|_{\mu} \leq \frac{\|k\|_{\mu}}{\mu} \frac{e^{\mu}-1}{\Gamma(\alpha+\beta)}$$

Suppose that $0 \leq t_1 \leq t_2 \leq 1$. Then we have

$$\left|B_{y(t_2)} - B_{y(t_1)}\right| \leq \frac{\left|{t_2}^{\beta} - {t_1}^{\beta}\right|}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big) d\tau$$

As $t_1 \to t_2$ independently, we conclude that

$$\left|B_{y(t_2)} - B_{y(t_1)}\right| \to 0 \text{ since } y \in B_{\epsilon}.$$

This shows that the operator B is relatively Compact on B_{ϵ}. Thus, by the Arzela Ascoli theorem[17], we conclude that B is compact on B_{ϵ}. By the Krasnoselskii fixed point theorem[17, 18], it follows that the initial value problem (1-5) has at least one solution on B_{ϵ}.

Theorem 6. (Main Result: Uniqueness of the Solution)

Suppose that $f: [0,1] \times R^3 \to R$ is a continuous function satisfying

$$\left|f\big(t, x_1, x_2, x_3\big) - f\big(t, y_1, y_2, y_3\big)\right|$$

$$\leq \sigma(t)(|x_1-y_1| + |x_2-y_2| + |x_3-y_3|)$$

$$\forall\, t \in [0,1], \qquad x_1, x_2, x_3, y_1, y_2, y_3 \in R,$$

$$\sigma(t) \in (L^1[0,1]; [0,\infty))$$

Then there exists a unique solution for the problem (1-5) for

$$r_1 < 1, \qquad r_1 = 2(1+\phi^*+\psi^*)\sigma^* \left[\frac{1}{\Gamma(\alpha+\beta)}\right]$$

where

$$\sigma^* = \int_0^1 \sigma(t) dt$$

Proof. Define an operator A on X by

$$A[x(t)] = \frac{1}{\Gamma(\alpha+\beta)} \times \int_0^t (t-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big) d\tau - \frac{t^{\beta}}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big) d\tau$$

Let us denote $sup_{0\leq t\leq 1}|f(t,0,0,0)| = f_0$,

and consider the sphere $B_r = \{x \in X: \|x\| \leq r\}$ where

$$r > \left(\frac{r_2}{1-r_1}\right), \qquad r_2 = \frac{2f_0}{\Gamma(\alpha+\beta)}$$

For each $t \in [0,1] and\ x \in B_r$, we have

$$|A[x(t)]| \leq \frac{1}{\Gamma(\alpha+\beta)} \times \int_0^t (t-\tau)^{\alpha+\beta-1} \left|f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big)\right| d\tau + \frac{t^{\beta}}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-\tau)^{\alpha+\beta-1} \left|f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big)\right| d\tau$$

$$\leq \frac{1}{\Gamma(\alpha+\beta)} \times \int_0^t (t-\tau)^{\alpha+\beta-1} \{\left|f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big) - f(\tau,0,0,0)\right| + |f(\tau,0,0,0)|\} d\tau + \frac{t^{\beta}}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-\tau)^{\alpha+\beta-1} \{\left|f\big(\tau, x(\tau), \phi x(\tau), \psi x(\tau)\big) - f(\tau,0,0,0)\right| + |f(\tau,0,0,0)|\} d\tau$$

$$\leq \frac{1}{\Gamma(\alpha+\beta)} \times \int_0^t (t-\tau)^{\alpha+\beta-1} \{\sigma(\tau)(|x(\tau)| + |\phi x| + |\psi x|) - f_0\} d\tau + \frac{t^{\beta}}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-\tau)^{\alpha+\beta-1} (\sigma(\tau)(|x| + |\phi x| + |\psi x|) + f_0) d\tau$$

$$\leq \frac{(1+\phi^*+\psi^*)\|x\|}{\Gamma(\alpha+\beta)}\int_0^1 \sigma(\tau)d\tau$$

$$+\frac{f_0}{\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1}\, d\tau$$

$$+\frac{(1+\phi^*+\psi^*)\|x\|}{\Gamma(\alpha+\beta)}\int_0^1 \sigma(\tau)d\tau+\frac{f_0}{\Gamma(\alpha+\beta)}$$

$$\leq \frac{(1+\phi^*+\psi^*)\|x\|}{\Gamma(\alpha+\beta)}\sigma^*$$

$$+\frac{f_0}{\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1}\, d\tau$$

$$+\frac{(1+\phi^*+\psi^*)\|x\|\sigma^*}{\Gamma(\alpha+\beta)}+\frac{f_0}{\Gamma(\alpha+\beta)}$$

$$\leq \frac{2(1+\phi^*+\psi^*)\|x\|}{\Gamma(\alpha+\beta)}\sigma^*+\frac{2f_0}{\Gamma(\alpha+\beta)}$$

This indicates that $\|A[x(t)]\| \leq r$.

Therefore,$FA \subseteq B_r$. Now we show that A is a contraction mapping,

For $x, y \in B_r$, we have

$$|Ax(t)-Ay(t)|$$

$$\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1}\big|f\big(\tau,x(\tau),\phi x(\tau),\psi x(\tau)\big) - f\big(\tau,y(\tau),\phi y(\tau),\psi y(\tau)\big)\big|d\tau$$

$$+\frac{t^\beta}{\Gamma(\alpha+\beta)}\int_0^1 (t-\tau)^{\alpha+\beta-1}\big|f\big(\tau,x(\tau),\phi x(\tau),\psi x(\tau)\big) - f\big(\tau,y(\tau),\phi y(\tau),\psi y(\tau)\big)\big|d\tau$$

$$\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-\tau)^{\alpha+\beta-1}\sigma(\tau)\{|x(\tau)-y(\tau)|$$

$$+|\phi x(\tau)-\phi y(\tau)|+|\psi x(\tau)-\psi y(\tau)|\}d\tau$$

$$+\frac{t^\beta}{\Gamma(\alpha+\beta)}\int_0^1 (t-\tau)^{\alpha+\beta-1}\sigma(\tau)\{|x(\tau)-y(\tau)|$$

$$+|\phi x(\tau)-\phi y(\tau)|+|\psi x(\tau)-\psi y(\tau)|\}d\tau$$

$$\leq \frac{(1+\phi^*+\psi^*)\sigma^*\|x-y\|}{\Gamma(\alpha+\beta)}$$

$$+\frac{(1+\phi^*+\psi^*)\sigma^*\|x-y\|}{\Gamma(\alpha+\beta)}$$

$$\leq \frac{2(1+\phi^*+\psi^*)\sigma^*\|x-y\|}{\Gamma(\alpha+\beta)}$$

Since $r_1 < 1$, it follows that A is a contraction. This proves that the system (1-5) has a unique solution.

4. ILLUSTRATIVE EXAMPLES

From theorems 5 and 6, we have proved the existence and the uniqueness of the solutions to the initial value problem (1-5). Now in this section, we will present here a couple of examples that support our results.

Example1. Consider the initial value problem

$$D^{\frac{1}{4}}[D^{\frac{3}{4}}]x(t) = \frac{t^3}{400}\left[\frac{|x(t)e^{-t}|}{1+|x(t)|}\right] + \int_0^t \frac{(t+\tau)^3|x(\tau)|[\cos\tau+\sin\tau]}{400[1+|x(\tau)|]} \quad . . . \ (6)$$

under the initial conditions $x(0)=x(1)=0,\ t \in [0,1]$

Here $\alpha=\frac{1}{4}, \beta=\frac{3}{4}$ and

$$f(t,x,y,z) = \frac{t^3}{400}\left[\frac{|x(t)|e^{-t}}{1+|x(t)|}+\frac{|y(t)|\cos \mathrm{t}}{1+|y(t)|}+\frac{|z(t)|\sin \mathrm{t}}{1+|z(t)|}\right]$$

$$\lambda(t,s) = \delta(t,s) = \frac{(t+s)^3}{400}, \qquad \sigma(t) = \frac{t^3}{400},$$
$$k(t) = \frac{3t^3}{400}$$

From the definitions of ϕ^* and ψ^*, it can be easily verified that

$$\phi^* = \psi^* = \frac{15}{1600}, \qquad \sigma^* = \frac{1}{1600}$$

Hence by existence theorem 5, we conclude that the initial value problem (6) has at least one solution.

Example2. Consider the initial value problem

$$D^{\frac{1}{4}}\left(D^{\frac{3}{4}}\right)x(t) = \frac{t^2}{200}\left[\frac{1}{1+|x(t)|}+\frac{1}{100}\int_0^t t^4\tau^3 x(\tau)\, d\tau\right]$$

under the conditions $x(0)=x(1)=0,\ t \in [0,1]$

Here

$$\alpha = \frac{1}{4}, \qquad \beta = \frac{3}{4}$$

Also,

$$f(t,x,y,z) = \frac{t^2}{200}\left[\frac{1}{1+|x(t)|} + |y(t)| + |z(t)|\right]$$

$$\lambda(t,\tau) = \delta(t,\tau) = \frac{t^4\tau^3}{200}, \qquad \sigma(t) = \frac{t^2}{200}$$

Using the definitions of ϕ^*, ψ^* and σ^*, it is clear that

$$\phi^* = \psi^* = \frac{1}{800}, \qquad \sigma^* = \frac{1}{600}$$

Calculating the value of r_1, we have

$$r_1 = 2(1 + \phi^* + \psi^*)\sigma^*\left[\frac{1}{\Gamma(\alpha+\beta)}\right]$$

This gives $r_1 \approx 0.0033$. By theorem 6, we conclude that the initial value problem (7) has a unique solution.

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